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# THE PROPER GENERALIZED DECOMPOSITION WITH APPLICATION IN PROBLEMS OF DYNAMICS

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**Abstract** *In this paper we address the new reduction method called Proper Generalized Decomposition (PGD) which is a discretization technique based on the use of separated representation of the unknown fields, specially well suited for solving multidimensional parametric equations. In this case, it is applied to the solution of dynamics problems. We will focus on the dynamic analysis of an one-dimensional rod with a unit harmonic load of frequency (ω) applied at a point of interest. In what follows, we will present the application of the methodology PGD to the problem in order to approximate the displacement field as the sum of the separated functions. We will consider as new variables of the problem, parameters models associated with the characteristic of the materials, in addition to the frequency. Finally, the quality of the results will be assessed based on an example.*

## 1. INTRODUCTION

Nowadays in spite of the impressive advances achieved in mechanical modeling, numerical analysis, discretization techniques and computer science in the last decade, there are many problems in science and engineering that still remain intractable due to their numerical complexity. Some of which will be impossible to solve by traditional methods. Of interests specifically, is the numerical simulation. The problem is that the numerical simulation cannot always provide the solutions for all points and at all times, because of that we enter in the discretization.

Authors have recently proposed a powerful new discretization technique based on the use of separated representations called Proper Generalized Decomposition (PGD), which demonstrates their ability to solve multidimensional models. This technique works by building separated representations of the solution, in that way, the complexity scales of the solution are linear with the dimension of the space in which the model is defined, instead of exponentially growing complexity of the mesh, based on commons discretization techniques. The PGD methods allows the efficient solution of the models to be defined in multidimensional spaces such as those found in quantum chemistry, kinetic theory descriptions of complex fluids, genetics, etc. For problems where the norms are defined in space and time, we add new coordinates resulting numerous possibilities.

The method consists of introducing unknown fields of the problem as extra coordinates, in the same way that the coordinates of time and space were originally defined in the problem. The problem is then solved once for different values of the variables that are unknown. This method can be generalized to case with extras coordinates as source terms or initial conditions.

Nevertheless, the models defined in high dimensional spaces suffer from the so-called curse of dimensionality. If we proceed to the solution of a model defined on N-dimensional space using a mesh technique based on discretization technique, where M nodes are used to discretize each coordinate space, the resulting number of nodes reaches the value of $M^N$. With M = 1000 and N = 30 (a very simple model), the numerical result would be $10^{90}$. It is important to remember that $10^{80}$ is the assumed number of elementary particles in the universe. Therefore, this new framework of PGD allows such for very efficient calculations for the problem of high dimensions because, in reality, the result has been previously calculated separately, so that post-processing of the solution is relatively simple.

In this case, if P is the number of iterations required to calculate the sum of the method, N the number of coordinates unknown and M the nodes to discretize each coordinate, the total number of PGD unknowns would be P x N x M instead $M^N$ of used in the meshing, based on usual discretization techniques, thus avoiding the curse of dimensionality. Hence, the PGD is a very powerful tool in numerical methods because it resolves previously unsolvable models.

## 2. PROPER GENERALIZED DECOMPOSITION

The models involving many fields or many parameters limit the use of standard discretization techniques due to the multidimensionality of the resulting model. Both the

need to use very fine meshes to represent the solutions prevents its integration and the resulting storage. One possibility to avoid the difficulties associated with the curse of dimensionality is the use of Proper Generalized Decomposition as discussed previously.

Such separated representation of the solution is not entirely new. Many years ago, a similar technique had already been developed for use in the field of quantum chemistry. In the 1980s, Pierre Ladeveze proposed this decomposition as an ingredient in the numerical method LATIN (Large Time Increment Method). In this era Ladeveze sought an efficient method to solve nonlinear models, but to accelerate the solution of the method, he needed this new technique, for that reason he used a separated solution in space-time [10].

$$u(x,t) \approx \sum_{i=1}^{i=N} X_i(x)\ .\ T_i(t) \tag{1}$$

This is what we define, in present day, as PGD in space-time. In what follows we will detail the recent advances of the solution of multidimensional models by applying the Proper Generalized Decomposition. This new method operates by a multidimensional generic function $u(x_1, x_2, \cdots x_Q)$ where Q is the number of dimensional spaces, ie $x_i$ denotes any usual coordinates (scalar or vectorial) including space and time and any boundary conditions or material parameters.

Therefore, separated representation of the function $u$ can be written as:

$$u(x_1, \cdots x_Q) \approx \sum_{i=1}^{i=N} F_i^1(x_1)\ \mathrm{x}\ \cdots\ \mathrm{x}\ F_i^Q(x_Q) \tag{2}$$

Where $(\mathrm{x}_1, \cdots \mathrm{x}_\mathrm{Q}) \in (\Omega_1\ \mathrm{x}\ \cdots \mathrm{x}\ \Omega_\mathrm{Q})$. The PGD approximation is thus a sum of N functional products, each involving a number of functions $\mathrm{F}_\mathrm{i}^\mathrm{j}(\mathrm{x}_\mathrm{j})$. Is important to know that the above functions are not known a priori but are calculated by the same method by introducing the separated representation of the solution in the weak formulation of the problem resulting in a nonlinear problem. This implies that iterations are needed at each enrichment step.

As is apparent from the above, the complexity of the solution scales linearly with the dimension of the space in which the model is defined, in contrast to the mentioned exponential evolution of the common methods.

Let us consider in more detail a numerical example applied to a dynamic system.

## 3. DYNAMIC EQUATION OF ONE DIMENSIONAL ROD

### 3.1. Strong and weak formulation

The strong formulation of dynamics of one dimensional rod can be written as:

$$\rho A \frac{\partial^2 U(x,t)}{\partial t^2} = EA \frac{\partial^2 U(x,t)}{\partial x^2} + f(x,\ t) \tag{3}$$

Where $u$ is the displacement field, $G$ is the stiffness, $\rho$ the density, A the area, which are assumed constant, and $f$ is a source term. Assume that the medium is excited harmonically with frequency ω and consequently the solution of shear wave propagation equation is also harmonic and its expression is $\mathrm{u}(\mathrm{x}, t) = u(x)e^{i\omega \mathrm{t}}$. Thus:

$$-\rho A\omega^2\, U(x) = EA\frac{\partial^2 U(x)}{\partial x^2} \tag{4}$$

In the case of a semi-infinite rod, excited at its boundary, one of the terms can be dropped since motion propagates from the boundary towards the infinite end. Considering a rightward motion, the particle velocity at any position may be obtained as: $\mathrm{v(x,t)} = \frac{\partial \mathrm{u}}{\partial \mathrm{t}} = -c_p \mathrm{f}'(\mathrm{x} - \mathrm{c_p t})$. Therefore, stresses may be linked with particle velocity since:

$$\sigma(x,t) = EA\frac{\partial U(x,t)}{\partial x} = E.f'(x - c_p t) = -\frac{E}{c_p}\, v(x,t) = -c_p.\rho.v(x,t) \tag{5}$$

This relationship indicates that, for the purposes of modeling, the semi-infinite portion may be substituted with an element providing stresses proportional to the particle velocity: a viscoelastic damper. This idea was first proposed by Lysmer and Kuhlemayer (1969) and is often used when defining Non-Reflecting Boundary Conditions (NRBC) in the context of Finite Element Modeling.

We consider a finite rod of length L with a load applied in one end, and non-reflecting boundary condition at the other. Therefore:

$$EA\left.\frac{\partial U(x)}{\partial x}\right|_{x=0} = c_p.\rho.A.i.\omega.U(0) \tag{6}$$

$$EA\left.\frac{\partial U(x)}{\partial x}\right|_{x=L} = 1 \tag{7}$$

For simple harmonic motion, damping can conveniently be expressed by using the concept of complex stiffness. It is useful taking account of the damping, particularly when the analysis is carried out in the frequency domain. This is usually written as:

$$\acute{E} = E + i\omega E' = E(1 + i2\zeta) \tag{8}$$

Defining $\eta = 2\zeta$, yields:

$$\acute{E} = E(1 + i\eta) \tag{9}$$

The strong formulation is:

$$\rho\omega^2 U(x) + E\frac{\partial^2 U(x)}{\partial x^2} + i\omega E'\frac{\partial^2 U(x)}{\partial x^2} \tag{10}$$

The weak formulation related to the equation (10) may be obtained by considering a trial function $U^*$ defined on the domain $\Omega$, and verifying the boundary conditions (6) and (7). The weak problem (after integration by parts) would read:

$$\int_\Omega U^*(x)\rho\omega^2\, U(x) - \int_\Omega \frac{\partial U^*(x)}{dx}.\acute{E}.\frac{\partial U(x)}{dx} + U^*(L) - U^*(0)i\rho c_p\omega U(0) = 0 \tag{11}$$

Substituting equation (9) in eq. (11) can be written as:

$$\int_\Omega U^*(x)\rho\omega^2\, U(x) - \int_\Omega \frac{\partial U^*(x)}{dx}.E(1 + i\eta).\frac{\partial U(x)}{dx} + U^*(L) - U^*(0)i\rho c_p\omega U(0) = 0 \tag{12}$$

## 4. PROPER GENERALIZED DECOMPOSITION APPLIED IN DYNAMIC

In what follows we are going to introduce the construction of Proper Generalized Decomposition applied to dynamic systems analyzed above. Because the damping and stiffness are considered unknowns, are assumed as new coordinates defined in the intervals. Thus, instead of solving for different values of the parameters, we prefer to introduce it as new coordinates. The price to be paid is the increase of the model dimensionality; however, as the complexity of PGD scales linearly with the space dimension the consideration of the stiffness as a new coordinate allows faster and cheaper solutions.

The PGD solution is sought in the form:

$$U \approx \sum_{i=1}^{N} X_i(x) Y_i(\omega) Z_i(E)\, V_i(\eta) \tag{13}$$

Hence, at the present iteration, n, we assume:

$$U^n \approx \sum_{i=1}^{n} X_i(x) Y_i(\omega) Z_i(E) V_i(\eta) \tag{14}$$

We seek the following solution $X_{n+1}, Y_{n+1}, Z_{n+1}$ and $V_{n+1}$ in order to simplify the process we are going to call it R (x), S (ω), T (E) and W (η). Therefore, the solution at enrichment step n + 1 can be written as:

$$U(x,\omega,E,\eta) \approx \sum_{i=1}^{i=n} X_i(x) Y_i(\omega) Z_i(E) V_i(\eta) + R(x) S(\omega) T(E) W(\eta) \tag{15}$$

Supposing $U^n$ to be known, we can compute the next iteration R (x), S (ω), T (E) and W (η). Similarly we suppose sequentially that S (ω), T (E) and W (η) are known of the previously iteration, and proceed to compute R (x). With the new value of R (x), T (E) and W (η) previously, can obtain S (ω), then with both of them values R (x) and S (ω) updated, and W(η) previously, can be determined T (E). Finally we can determine W (η) with previous values updated. The process is repeated in a suitable fixed-point iteration scheme, until reaching a state of convergences, where the results will be the new products $X_{n+1}$, $Y_{n+1}$, $Z_{n+1}$ and $V_{n+1}$ We illustrated each steps as follows:

### 4.1. Computing R(x) from S (ω), T (E) and W (η)

Considering the weak problem of equation (12), the weighting function $U^*$ is then assumed as:

$$U^* = (R,S,T,W)^* = R^*\, S\, T\, W + R\, S^*\, T\, W + R\, S\, T^*\, W + R\, S\, T\, W^* \tag{16}$$

We suppose that S (ω), T (E) and W (η) are known from the previous interaction and compute an update for R (x). In this case the trial function is:

$$U^* = (R,S,T,W)^* = R^*(x) S(\omega) T(E) W(\eta) \tag{17}$$

Introducing the equations (17) and (15) into (12) its result:

$$\iiiint_{\Omega * \Im * \aleph * I} R^* STW\, \rho A \omega^2 \left(\sum_{i=1}^{i=n} X_i Y_i Z_i V_i + RSTW\right) dx d\omega dE d\eta -$$

$$\iiiint_{\Omega * \Im * \aleph * I} R^{*\prime} STW\, E(1+i\eta) A. \left(\sum_{i=1}^{i=n} X_i' Y_i Z_i V_i + R' STW\right) dx d\omega dE d\eta +$$

$$\iiint_{\Im*\aleph*I} R^*(L)STW\ d\omega dEd\eth -$$

$$\iiint_{\Im*\aleph*I} R^*(0)STW.\, i\rho A\omega c_p \left(\sum_{i=1}^{i=n} X_i(0)Y_iZ_iV_i + R(0)STW\right) d\omega dEd\eth = 0 \quad (18)$$

The integrals are performed in the $\Im\ x\ \aleph\ x\ I$ domain. Thus, by using the following notations:

$$\begin{cases} \iiint_{\Im*\aleph*I} STW\rho A\omega^2 Y_iZ_iV_i\ d\omega dEd\eth = \alpha_i \\ \iiint_{\Im*\aleph*I} STW\ E(1+i\eth)A\ Y_iZ_iV_i\ d\omega dEd\eth = \beta_i \\ \iiint_{\Im*\aleph*I} STW\ \omega A\sqrt{\rho E}\ Y_iZ_iV_i\ d\omega dEd\eth = \delta_i \end{cases} \begin{cases} \iiint_{\Im*\aleph*I} (STW)^2\rho A\omega^2\ d\omega dEd\eth = \gamma_1 \\ \iiint_{\Im*\aleph*I} (STW)^2 E(1+i\eth)A\ d\omega dEd\eth = \gamma_2 \\ \iiint_{\Im*\aleph*I} STW\ \ d\omega dEd\eth = \emptyset_1 \\ \iiint_{\Im*\aleph*I} (STW)^2\ \omega A\sqrt{\rho E}\ \ d\omega dEd\eth = \emptyset_2 \end{cases} \quad (19)$$

The celerity is expressed in the terms of stiffness $c = \sqrt{E/\rho}$. Substituting (19) in eq. (18) reduces to:

$$\int_\Omega R^*\gamma_1 R\ dx + \int_\Omega R^* \sum_{i=1}^{i=n} \alpha_i X_i\ dx - \int_\Omega R^{*'} \sum_{i=1}^{i=n} \beta_i X_i'\ dx - \int_\Omega R^{*'} \gamma_2 R'\ dx +$$

$$R^*(L)\emptyset_1 - R^*(0)\sum_{i=1}^{i=n} i\delta_i X_i(0) - R^*(0)i\,\emptyset_2 R(0) = 0 \quad (20)$$

This can be grouped as:

$$\int_\Omega R^*\gamma_1 R\ dx - \int_\Omega R^{*'} \gamma_2 R'\ dx + \int_\Omega R^* \left(\sum_{i=1}^{i=n} \alpha_i X_i + \beta_i X_i''\right) dx +$$

$$R^*(L)\left(\emptyset_1 - \sum_{i=1}^{i=n} \beta_i X_i'(L)\right) + R^*(0)\sum_{i=1}^{i=n}\left(\beta_i X_i'(0) - X_i(0)i\delta_i\right) + R^*(0)i\,\emptyset_2 R(0) = 0 \quad (21)$$

This equation can be interpreted as weak formulation of the dynamics of a rod, assuming an equivalent excitation frequency $\omega_{eq} = 1$, the equivalent density $\rho_{eq} = \gamma_1$ and Young modulus $E_{eq}' = \gamma_2$, under an external loading $f_{ext}$ defined in terms of $X_i$ and its second derivative, two tips forces $f_{tip}$ which are also defined in terms of $X_i$, and finally a non-reflecting boundary condition, defined by a viscous damping depend on the considered equivalent frequency and density:

$$\begin{cases} f_{ext}(x) = \sum_{i=1}^{i=n}\left(\alpha_i X_i + \beta_i X_i''\right) \\ f_{tip}(x=0) = \sum_{i=1}^{i=n}\left(\beta_i X_i'(0) - X_i(0)i\delta_i\right) \\ f_{tip}(x=L) = \emptyset_1 - \sum_{i=1}^{i=n} \beta_i X_i'(L) \\ (\omega\rho c)_{eq} = \emptyset_2 \end{cases} \quad (22)$$

That can be solved by using any suitable discretization techniques.

### 4.2. Computing S (ω) from R (x), T (E) and W (ŋ)

From the update R (x) and the previously used T (E) and W (ŋ) we can update S (ω). Therefore, the test function is written as:

$$U^* = (R,S,T,W)^* = R(x)S^*(\omega)T(E)\mathrm{W}(\eta) \tag{23}$$

And the weak formulation becomes:

$$\iiiint_{\Omega*\Im*\aleph*I} RS^*TW\ \rho A\omega^2\ \left(\sum_{i=1}^{i=n} X_iY_iZ_iV_i + RSTW\right) dxd\omega dEd\eta -$$
$$\iiiint_{\Omega*\Im*\aleph*I} R'S^*TW\ E(1+i\eta)A.\left(\sum_{i=1}^{i=n} X_i'Y_iZ_iV_i + R'STW\right) dxd\omega dEd\eta +$$
$$\iiint_{\Im*\aleph*I} R(L)S^*TW\ \ d\omega dEd\eta -$$
$$\iiint_{\Im*\aleph*I} R(0)S^*TW.\, i\rho A\omega c_p\ \left(\sum_{i=1}^{i=n} X_i(0)Y_iZ_iV_i + R(0)STW\right) d\omega dEd\eta = 0 \tag{24}$$

Integrating over $\Omega * \aleph * I$:

$$\begin{cases} \iiint_{\Omega*\aleph*I} RTW\rho AX_iZ_iV_i\ dxdEd\eta = \alpha_i \\ \iiint_{\Omega*\aleph*I} R'TW\ E(1+i\eta)AX_i'Z_iV_i dxdEd\eta = \beta_i \\ \iint_{\aleph*I} TW\ A\sqrt{\rho E}\ Z_iV_i\ dEd\eta = \delta_i \end{cases} \begin{cases} \iiint_{\Omega*\aleph*I} (RTW)^2\rho A\ dxdEd\eta = \gamma_1 \\ \iiint_{\Omega*\aleph*I} \left(R'TW\right)^2 E(1+i\eta)Adxd Ed\eta = \gamma_2 \\ \iint_{\aleph*I} TW\ \ dEd\eta = \emptyset_1 \\ \iint_{\aleph*I} (TW)^2\ A\sqrt{\rho E}\ \ dEd\eta = \emptyset_2 \end{cases} \tag{25}$$

Thus:

$$\int_{\Im}\ S^*\left(\gamma_1\omega^2 - \gamma_2 - i\omega\emptyset_2R^2(0)\right)S\ \ d\omega +$$
$$\int_{\Im}\ S^*\left[\emptyset_1R(L) + \sum_{i=1}^{i=n}(\alpha_i\omega^2 - \beta_i - i\omega X_i(0)R(0)\delta_i)Y_i\right] d\omega = 0 \tag{26}$$

The strong form reads:

$$\left(\gamma_1\omega^2 - \gamma_2 - i\omega\emptyset_2R^2(0)\right)S +$$
$$\emptyset_1R(L) + \sum_{i=1}^{i=n}(\alpha_i\omega^2 - \beta_i - i\omega X_i(0)R(0)\delta_i)Y_i = 0 \tag{27}$$

Since the equation is true for all $S^*$, S may be expressed as:

$$S = \frac{\emptyset_1R(L) + \sum_{i=1}^{i=n}(\alpha_i\omega^2 - \beta_i - i\omega X_i(0)R(0)\delta_i)Y_i}{\gamma_2 - \gamma_1\omega^2 + i\omega\emptyset_2R^2(0)} \tag{28}$$

### 4.3. Computing T (E) from R (x), S (ω) and W (η)

From the previously computed R (x) and S (ω), and W (η) previous, we can obtain T (E). The test function is now given by:

$$U^* = (R,S,T,W)^* = R(x)\ S(\omega)\ T^*(E)W(\eta) \tag{29}$$

The weak form reads:

$$\iiiint_{\Omega*\Im*\aleph*I} RST^*W\ \rho A\omega^2\ \left(\sum_{i=1}^{i=n} X_iY_iZ_iV_i + RSTW\right) dxd\omega dEd\eta -$$

$$\iiiint_{\Omega*\Im*\aleph*I} R'ST^*W\ E(1+i\eta)A.\left(\sum_{i=1}^{i=n} X_i'Y_iZ_iV_i + R'STW\right)dxd\omega dEd\eta + \iiint_{\Im*\aleph*I} R(L)ST^*W\ d\omega dEd\eta - \iiint_{\Im*\aleph*I} R(0)ST^*W.i\rho A\omega c_p\left(\sum_{i=1}^{i=n} X_i(0)Y_iZ_iV_i + R(0)STW\right)d\omega dEd\eta = 0 \quad (30)$$

Now, we integrate $\Omega * \Im * I$ domain:

$$\begin{cases} \iiint_{\Omega*\Im*I} RSW\rho A\omega^2 X_iY_iV_i\ dxd\omega d\eta = \alpha_i \\ \iiint_{\Omega*\Im*I} R'SW\ (1+i\eta)A\ X_i'Y_iV_i\ dxd\omega d\eta = \beta_i \\ \iint_{\Im*I} SW\ \omega A\ Y_iV_i\ d\omega d\eta = \delta_i \end{cases} \begin{cases} \iiint_{\Omega*\Im*I} (RSW)^2\rho A\omega^2\ dxd\omega d\eta = \gamma_1 \\ \iiint_{\Omega*\Im*I} \left(R'SW\right)^2(1+i\eta)A\ dxd\omega d\eta = \gamma_2 \\ \iint_{\Im*I} SW\ d\omega d\eta = \emptyset_1 \\ \iint_{\Im*I} (SW)^2 A\omega\ d\omega d\eta = \emptyset_2 \end{cases} \quad (31)$$

From the equation (30) and taking to account the notations in eq. (31), results:

$$\int_{\aleph} T^*\left(\gamma_1 - E\gamma_2 - i\rho c_p\emptyset_2R^2(0)\right)T\ dE + \int_{\aleph} T^*\left[\emptyset_1R(L) + \sum_{i=1}^{i=n}\left(\alpha_i - \beta_iE - i\rho c_pX_i(0)R(0)\delta_i\right)Z_i\right]dE = 0 \quad (32)$$

Coming back to the strong form, reads:

$$T = \frac{\emptyset_1R(L) + \sum_{i=1}^{i=n}\left(\alpha_i - E\beta_i - i\sqrt{\rho E}X_i(0)R(0)\delta_i\right)Z_i}{E\gamma_2 - \gamma_1 + i\sqrt{\rho E}\emptyset_2R^2(0)} \quad (33)$$

This expression allows obtaining an expression for T, since the condition should be true for all $T^*$. Wave celerity is expressed in terms of E in order to highlight this dependence.

### 4.4. Computing W (η) from R (x), S (ω) and T (E)

Finally with all the previous values update, the test function is now given by:

$$U^* = (R,S,T,W)^* = R(x)S(\omega)T(E)W^*(\eta) \quad (34)$$

The weak formulation reads:

$$\iiiint_{\Omega*\Im*\aleph*I} RSTW^*\ \rho A\omega^2\left(\sum_{i=1}^{i=n} X_iY_iZ_iV_i + RSTW\right)dxd\omega dEd\eta - \iiiint_{\Omega*\Im*\aleph*I} R'STW^*\ E(1+i\eta)A.\left(\sum_{i=1}^{i=n} X_i'Y_iZ_iV_i + R'STW\right)dxd\omega dEd\eta + \iiint_{\Im*\aleph*I} R(L)STW^*\ d\omega dEd\eta - \iiint_{\Im*\aleph*I} R(0)STW^*.i\rho A\omega c_p\left(\sum_{i=1}^{i=n} X_i(0)Y_iZ_iV_i + R(0)STW\right)d\omega dEd\eta = 0 \quad (35)$$

Can be integrated $\Omega * \Im * \aleph$ domains:

$$\begin{cases} \iiint_{\Omega*\Im*\aleph} RST\rho A\omega^2 X_i Y_i Z_i \, dxd\omega dE = \alpha_i \\ \iiint_{\Omega*\Im*\aleph} R'ST \; EA \; X_i' Y_i Z_i \, dxd\omega dE = \beta_i \\ \iint_{\Im*\aleph} ST \; \omega A\sqrt{\rho E} \; Y_i Z_i \, d\omega dE = \delta_i \end{cases} \quad \begin{cases} \iiint_{\Omega*\Im*\aleph} (RST)^2 \rho A\omega^2 \; dxd\omega dE = \gamma_1 \\ \iiint_{\Omega*\Im*\aleph} \left(R'ST\right)^2 EA \; dxd\omega dE = \gamma_2 \\ \iint_{\Im*\aleph} ST \; d\omega dE = \emptyset_1 \\ \iint_{\Im*\aleph} (ST)^2 \, A\omega\sqrt{\rho E} \; d\omega dE = \emptyset_2 \end{cases} \tag{36}$$

Equation (35) reads:

$$\int_I \; W^* (\gamma_1 - (1 + i\eta)\gamma_2 - i\emptyset_2 R^2(0))W \; d\eta +$$

$$\int_I \; W^* \left[\emptyset_1 R(L) + \sum_{i=1}^{i=n} (\alpha_i - \beta_i(1 + i\eta) - iX_i(0)R(0)\delta_i)V_i\right] d\eta = 0 \tag{37}$$

Coming back to strong form and since this equation is true for all $W^*$, W can be obtained:

$$W = \frac{\emptyset_1 R(L) + \sum_{i=1}^{i=n} (\alpha_i - \beta_i(1 + i\eta) - iX_i(0)R(0)\delta_i)V_i}{(1 + i\eta)\gamma_2 - \gamma_1 + i\emptyset_2 R^2(0)} \tag{38}$$

### 4.5. Convergence criterion

In the calculation of the new values of R (x), S (ω), T (E) and W (η), are called ¨new¨ to the present iteration and ¨old¨ the previous iteration. The convergence criterion used is the following:

$$\frac{|\alpha_{new} - \alpha_{old}|}{\alpha_{old}} < \epsilon_1 \tag{39}$$

Where:

$$\alpha_{new} = \alpha_r|_{new} \, . \; \alpha_s|_{new} \, . \; \alpha_t|_{new} \, . \alpha_w|_{new} \tag{40}$$

And:

$$\begin{aligned} \alpha_r|_{new} &= \int_\Omega \; R_{new} * conj \; (R_{new}) \, dx \\ \alpha_s|_{new} &= \int_\Im \; S_{new} * conj \; (S_{new}) \, d\omega \\ \alpha_t|_{new} &= \int_\aleph \; T_{new} * conj \; (T_{new}) \, dE \\ \alpha_w|_{new} &= \int_I \; W_{new} * conj \; (W_{new}) \, d\eta \end{aligned} \tag{41}$$

The same procedure to the case ¨old¨. After reaching convergence we write $X_{n+1}(x) = R(x)$, $Y_{n+1}(\omega) = S(\omega)$, $Z_{n+1}(E) = T(E)$ and $V_{n+1}(\eta) = W(\eta)$ . The enrichment procedure must continue until reaching the convergence of the enrichment global procedure at iteration *N*. The global convergence criteria used in this work is the following:

$$\alpha_r|_{new} \, . \; \alpha_s|_{new} \, . \; \alpha_t|_{new} \, . \alpha_w|_{new} < \epsilon_2 \tag{42}$$

Notice, also that the convergence velocity of the fixed-point process is high. Usually, only a maximum of 15 iterations is required to reach the convergence. Thus, the global convergence was reached with 8 functional couples.

## 5. NUMERICAL RESULTS

In this section we present the numerical example applied to the method Proper Generalized Decomposition to validate our approach and evaluating its efficiently. In order to illustrate the power of the method and the potential use for the computed solutions beyond optimization problems, we present some particular results.

The PGD method was used to approximate the solution of the displacement field U(x) for all combinations of the parameters (x, ω, E, η) in its respective ranges. Let us considerer a rod, 1 m long, with unitary transverse area and linear density. The unknown parameter E takes values in the range [0.9 x $10^4$ 1.1 x $10^4$] and η ∈ [0.1 0.2]. The rod is excited at the right end, and has non-reflecting boundary condition at the left one. The exciting force is variable between 10 and 110 Hz.

The comparisons between the predictions based on the PGD and those based on direct FEM calculations are presented in figure 1. The results were obtained for various properties of the materials and were performed for specific excitation frequencies. In Figures (b) and (c), we can observe that the deformations have the shapes of sin (π·x/L) and sin (2π·x/L) respectively. These deformations correspond to natural frequencies of 50 Hz and 105 Hz. As seen in the figures, the separated representation method PGD provides accurate results when compared with the results provided by the finite element method.

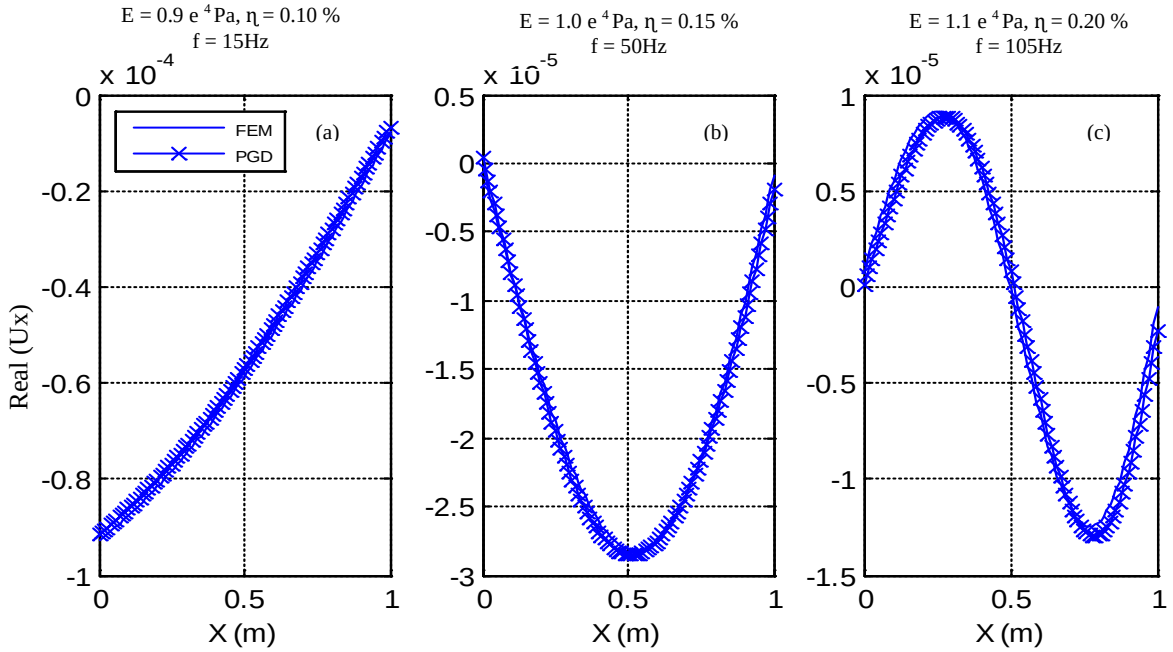


**Figure 1**. Displacement magnitudes obtained through a direct calculation against PGD approximation, for 3 combinations of complex and frequency.

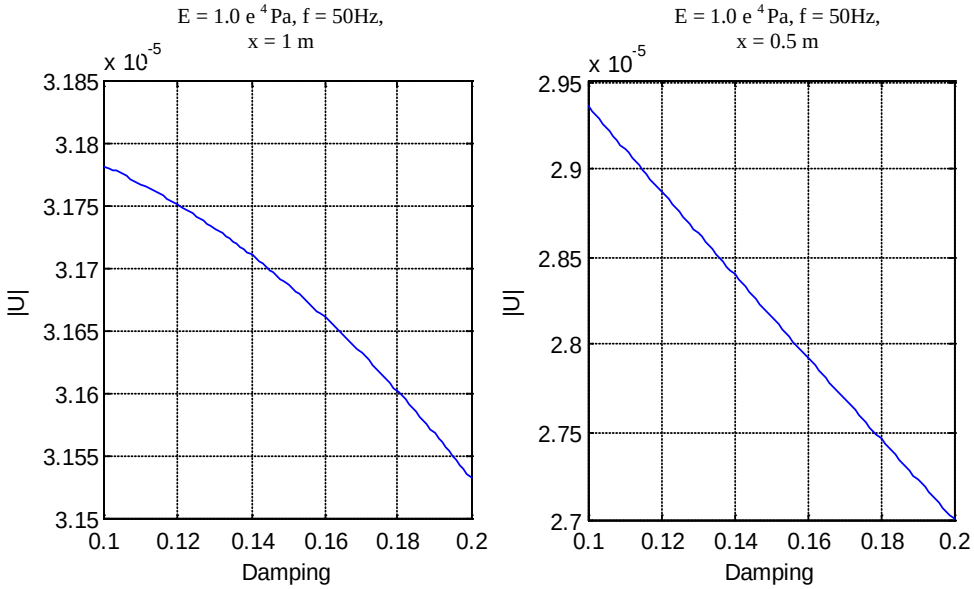


**Figure 2.** Displacement magnitudes obtained through PGD approximation for all damping at two positions: 1 m and 0.5 m.

Figure 2 presents the magnitude of the displacement of two specific points in the bar as a function of different values of damping. It can be seen that as the damping increases, the magnitude of the displacement decreases, as is normally the case.

The importance of each term is assessed with reference to figure 3, where the displacement magnitude is plotted as a function of the number of PGD terms used. These plots are presented considering two points of the space domain, and three excitation frequencies.

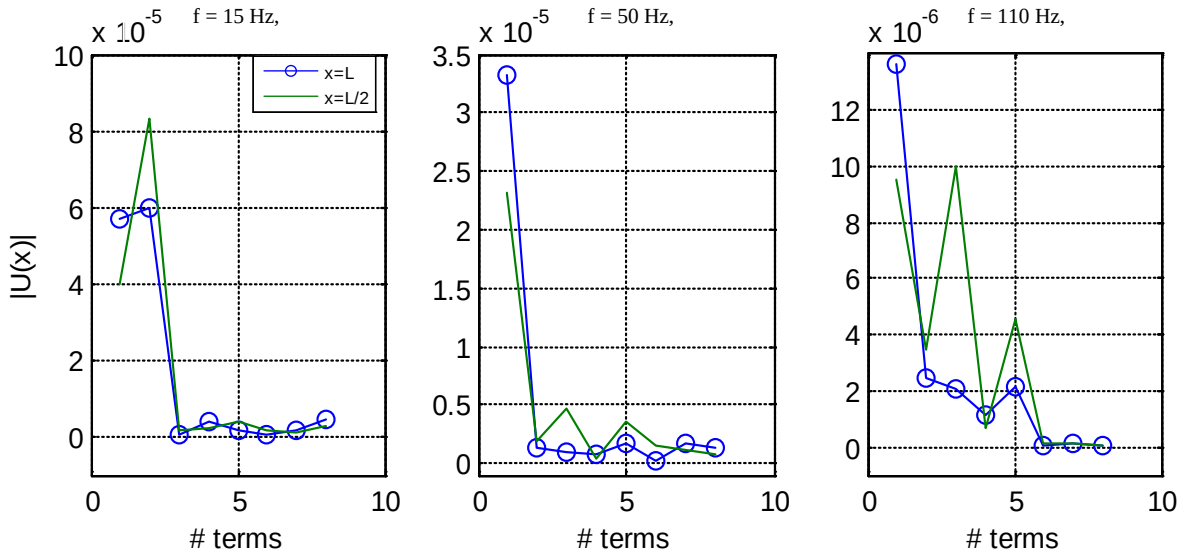


**Figure 3**. Displacement magnitudes as a function of the number of PGD terms used for 3 tip excitation frequencies.

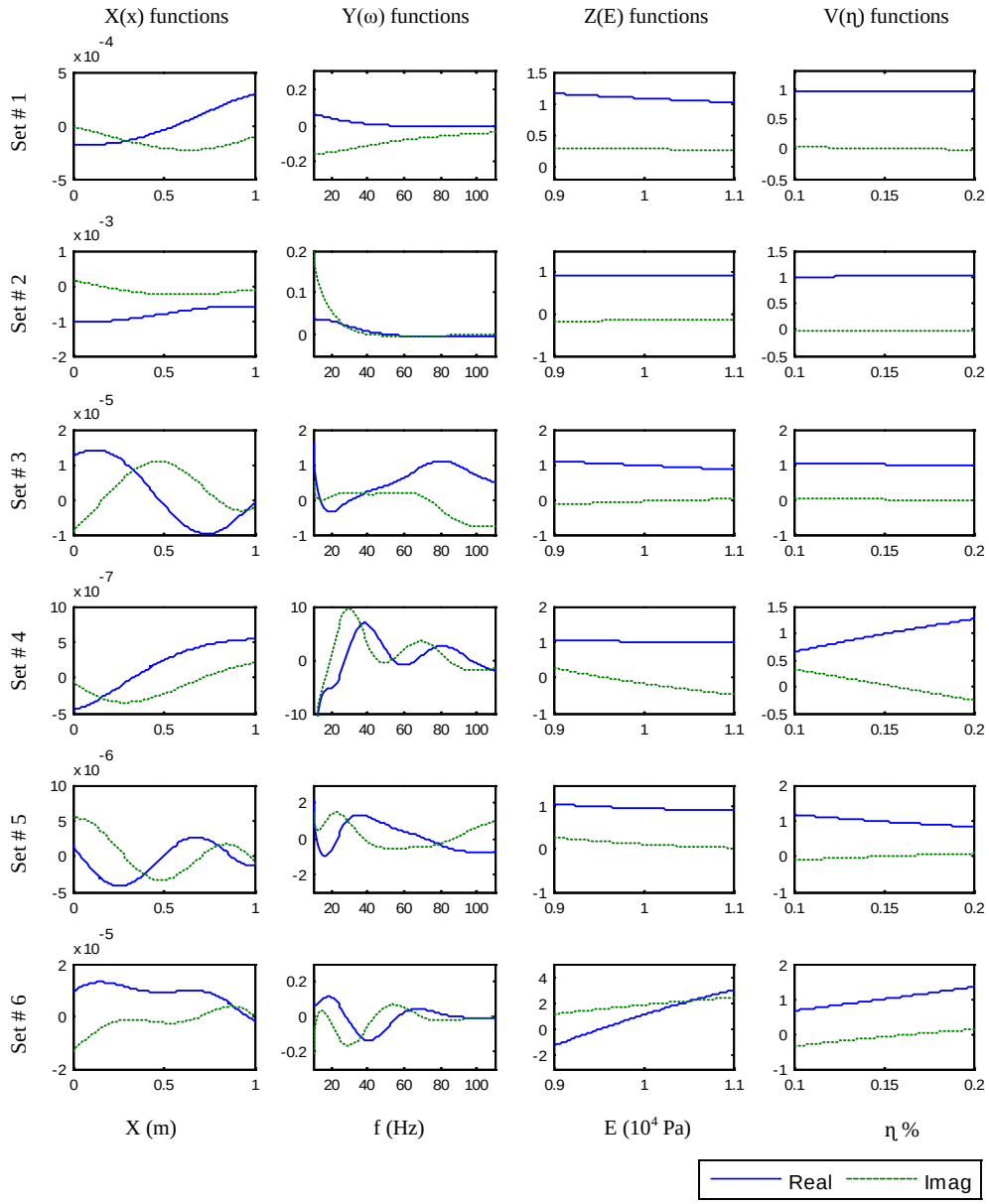


**Figure 4**. First six (out of 8) sets of 4 functions resulting from the case studied PGD. One row per set, and one column per variable.

PGD has been applied leading to decomposition in the form of 8 terms. The first six are presented in figure 4. The Fig. 5 depicts the evolution of the error with the number of terms N in the separated representation of U. This numerical error depends basically on the discretization.

Finally as we compute a variable-separated representation of the solution, the amount of memory required to store U is negligible compared to what would have been required to store it on an equivalent four-dimensional grid.

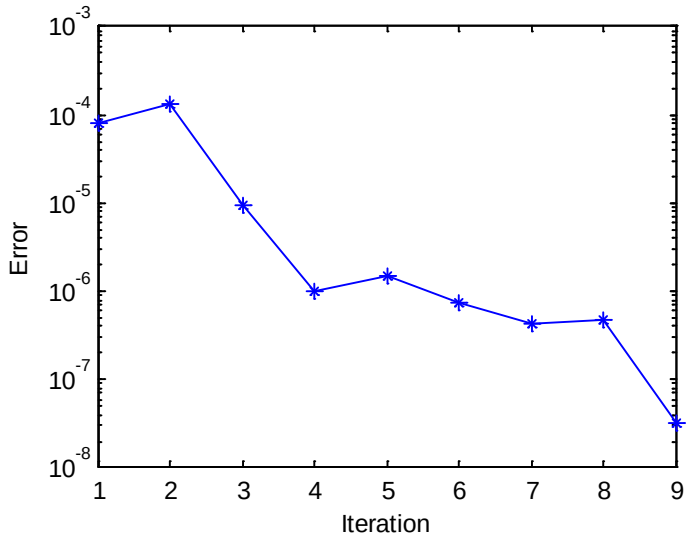


**Figure 5**. Error versus the number of sums in the finite sums decomposition.

## 6. CONCLUSIONS

In this work we have developed the new reduction method based on Proper Generalized Decomposition focusing in dynamic systems. We have shown that PGD is extremely suitable for high-dimensional models that have been defined. This is because it is based on separated representation of the solution which makes it possible to increase the size by introducing extra coordinates avoiding the exponential complexity on discretization methods commons or also called the curse of dimensionality.

Another advantage lies in the fact that some models involve unknown parameters which may be included as additional coordinates, ie the problem is solved just once, but for any value of the unknown parameter, playing the same role as the spatial coordinates or temporary, which can also be generalized to any additional coordinates as source terms, boundary conditions or initial conditions.

As we observed in the numerical results of the dynamic problem discussed above, the application of this method obtained accurate results compared with finite element methodology. To date there has not been an issue where the PGD method has not behaved satisfactorily in comparison to other discretization techniques. Furthermore, this offers a fast enough computer calculation. In this case the PGD allowed for computing general parametric solutions where the frequency and parameters materials were considered as extra coordinates.

Finally, this methodology has been found extremely efficient and exact in a wide variety of the problems, therefore, we can consider that the Proper Generalized Decomposition constitute a new paradigm in scientific computing and numerical analysis.

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